Addressing, with the concourse of CAS, diverse open issues dealing with the manifold and subtle concept of geometric locus

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Why Locus Matters in Mathematics Education

Locus computation is a characteristic, defining feature of Dynamic Geometry (DG) software:

> "There is a wide consensus among DG developers to consider locus computation as one of the five basic properties in the DG paradigm (together with dynamic transformation, measurement, free dragging and animation) (X. S. Gao, 1999)"*

From Sketchpad** (1963) to modern DG systems

- Already present over sixty years ago in early systems as *Sketchpad*
- More recently, the incorporation of symbolic engines into DG environments has raised expectations for more precise and interpretable results

(*) Abanades, M.; Botana, F.; Montes, A.; Recio, T. (2014). An algebraic taxonomy for locus computation in dynamic geometry. Computer-Aided Design 56 , 22-33. DOI: 10.1016/j.cad.2014.06.008

X. S. Gao, Automated geometry diagram construction and engineering geometry, in: X. S. Gao, D. Wang, L. Yang (Eds.), ADG 1998, Vol. 1669700 of Lecture Notes in Artificial Intelligence, Springer, Heidelberg, 1999, pp. 232–257

(**) Ivan Shutherland 1963, premio Turing 1988. **Not to be confused with The Geometer's Sketchpad**

Dynamic Geometry Today

Yet, current locus computation commands in many DG programs are either

just visual plots (mover---tracer)

just approximate, numerical equations

I.e. there is NOT a symbolic approach to locus computation in many DG programs.

On the other hand, GeoGebra uses symbolic computation for locus computation, but in the following way:

Visualization of Geometric Loci in GeoGebra

- GeoGebra Discovery first formulates the construction symbolically
- Computation of symbolic equations is internally straightforward
- Visualization necessarily requires a numerical specialization of these equations, followed by elimination

CAS as a Research Laboratory

Exploring algorithmic possibilities that could eventually benefit DG systems.

Focus on:

- Loci recognition
- Parameters and specialization
- Hypothesis discovery

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Recognition of a Locus Beyond Numerical Appearance

EXAMPLE 1: Given two fixed points A and B, determine the locus of points C such that the distance AC is twice the distance BC

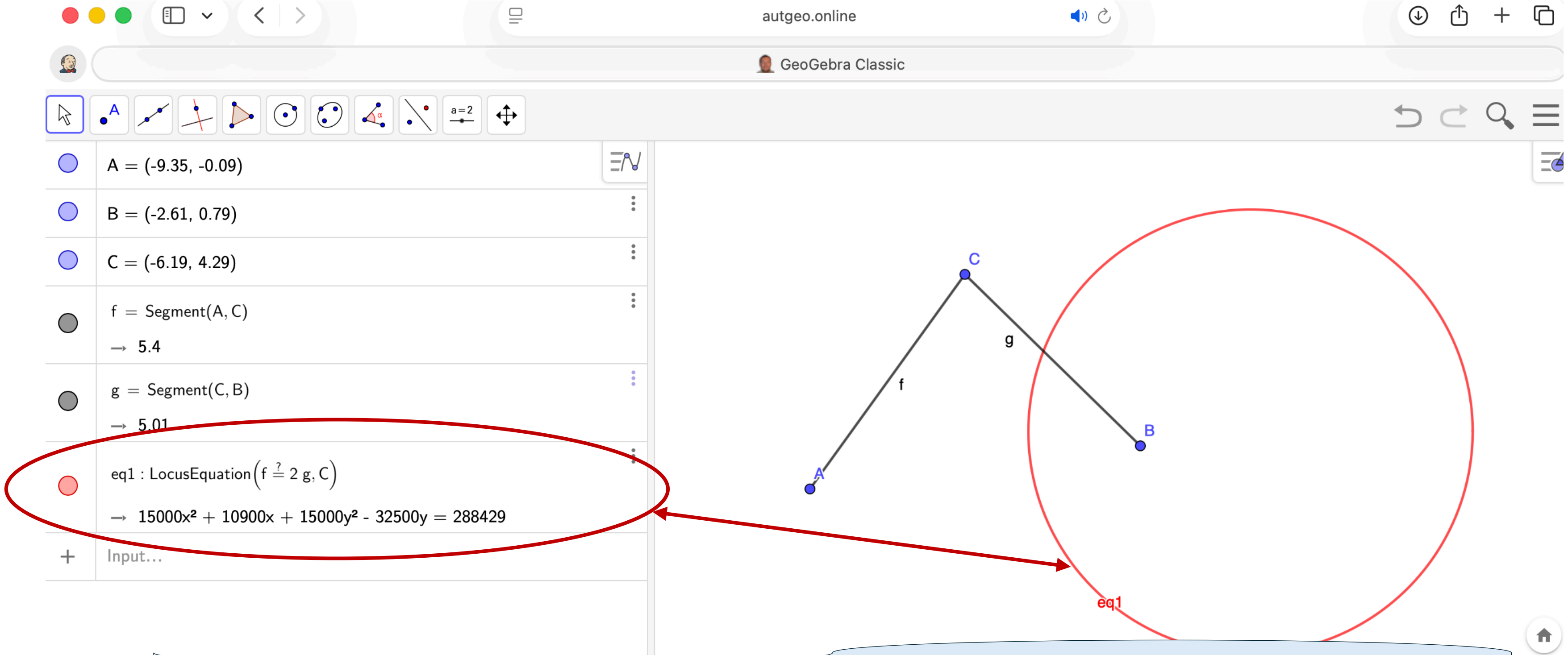


Figure 1: GeoGebra Discovery output of LocusEquation(f=2g,C)

- What is the exact geometric nature of the generic locus?
- How does it symbolically depend on A and B?
- Under what conditions is the locus complete or degenerate?

Recognition of a Locus Beyond Numerical Appearance

EXAMPLE 2: Determine the locus of points C such that, in the right triangle CAB, the square of the hypotenuse is equal to twice the product of the legs

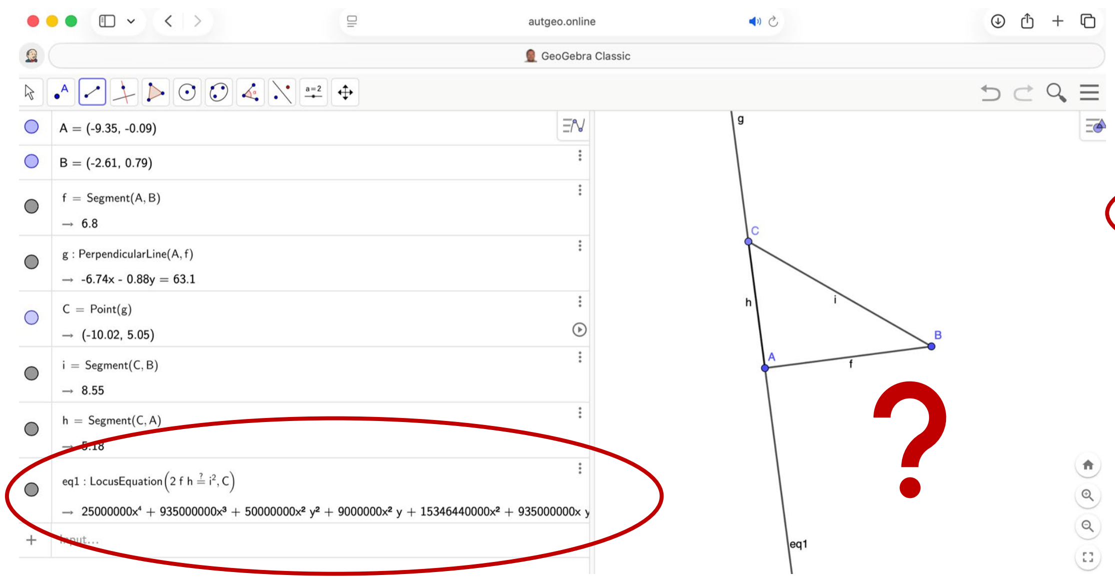


Figure 2: GeoGebra Discovery web version

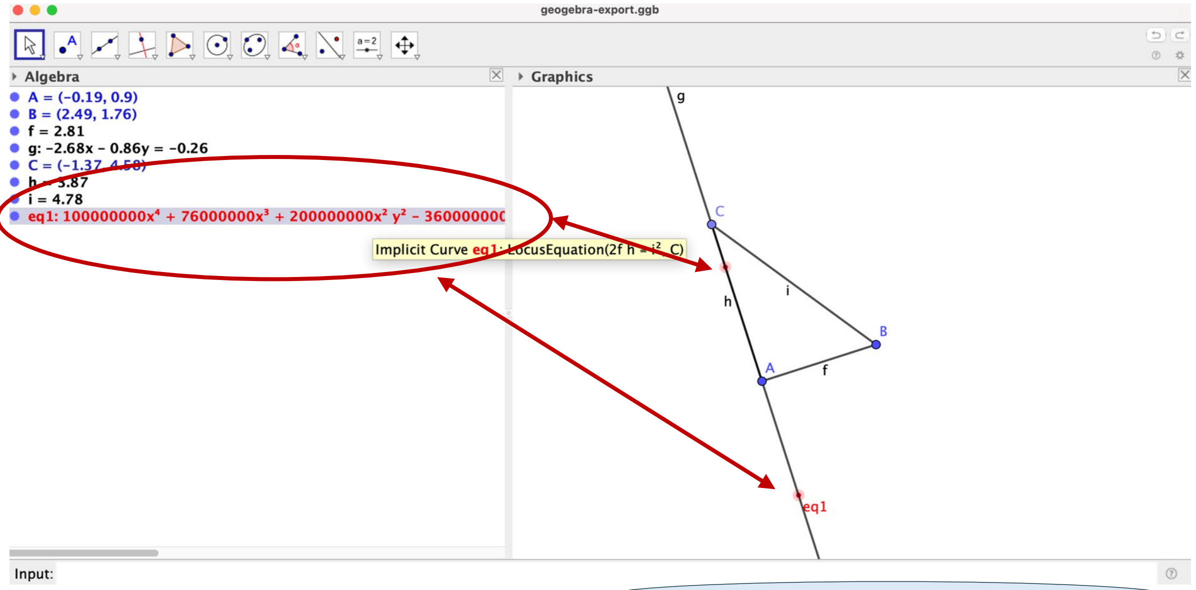


Figure 3: GeoGebra Discovery desktop version

- We obtain two points (red ones), but what is their relationship with A and B?
- Outputs algebraically correct but geometrically opaque

Educational Takeaway

Visualization ≠ Understanding

Visual output

↓

Interpretation

↓

Meaning

↓

Proof

Fundamental Questions

- How to deal with symbolic locus, i.e. with locus defined in the context of geometric constructions with arbitrary (parametric?) coordinates, and how to deal with the corresponding specialization for numerical values of the coordinates of the basic points of the same construction?
- Should we consider, in the EliminationIdeal protocol that is usually associated to locus computation, the parameters as variables or as elements of the field of coefficients?
- In what sense will we "discover" geometric statements by looking for the locus of some point in a construction that verifies a certain constraint?

Role of Independent Variables or Parameters

p = set of independent variables
y = set of locus point coordinates
x = set of extra variables (dependent)

$$
\begin{array}{ccc}
\boldsymbol{K[\mathrm{p,x,y}]} & \hookrightarrow & \boldsymbol{K(\mathrm{p})[\mathrm{x,y}]} \\
\mathfrak{a} & \longmapsto & \mathfrak{c}=\mathfrak{a}K(\mathrm{p})[\mathrm{x,y}] \\
\downarrow & & \downarrow \\
\boldsymbol{K[\mathrm{p,y}]} & \hookrightarrow & \boldsymbol{K(\mathrm{p})[\mathrm{y}]} \\
\mathfrak{b}=\mathfrak{a} \cap K[\mathrm{p,y}] & \longmapsto & \mathfrak{d}= \mathfrak{c} \cap K(\mathrm{p})[\mathrm{y}]=\mathfrak{b}^e = \mathfrak{b}K(\mathrm{p})[\mathrm{y}] \\
 & & \\
\mathfrak{d}^c & \longleftarrow\!\shortmid & \mathfrak{d}
\end{array}
$$

- **Lemma 2.3.**

$$\mathfrak{b}^e = \mathfrak{d}$$

Lemma 2.4. Under the assumption that none of the primary components of $\mathfrak{a}$ contain a non-zero polynomial in $K[\mathrm{p}]$

the generators of $\mathfrak{d}^c$ generate $\mathfrak{b}$

Dana-Picard, T., Recio T. : From Thales theorem to octic curves. Maple Trans., Vol. 4, No. 3, Article 18022. October 2024
https://doi.org/10.5206/mt.v4i3.18022

Specialization and Elimination

$p \in K^r$
$x \in K^t$
$y \in K^s$

$\mathcal{V}$= variety contained in K^{r+t+s}

$\mathcal{W}$= cylinder in K^{r+s} of the variety only in terms of p

$\Pi: K^{r+t+s} \longrightarrow K^{r+s}$ the projection $\Pi(p, x, y) = (p, y)$

$\mathcal{W}'$ cylinder in K^{r+t+s} defined by $\mathcal{W}$

Lemma.

$$\Pi(\mathcal{V} \cap \mathcal{W}') = \Pi(\mathcal{V}) \cap \mathcal{W}$$

Corolario.

$$\mathcal{V}\left(Elim_{(p,y)}(\mathcal{V} + \mathcal{W}')\right) = \overline{\Pi(\mathcal{V} \cap \mathcal{W}')} \subseteq \overline{\Pi(\mathcal{V})} \cap \mathcal{W} = \mathcal{V}\left(Elim_{(p,y)}(\mathcal{V})\right) \cap \mathcal{W}$$

The set of differences is in

$$\left(\overline{\Pi(\mathcal{V})} - \Pi(\mathcal{V})\right) \cap \mathcal{W}$$

A Challenging Geometric Construction

EXAMPLE 3: Given a triangle ABC, a point F and a free point P. The feet of the lines AF, BF and CF are Ac, Bc, Cc, respectively. The points A, B and C are mirrored at P; A',B' and C', respectively

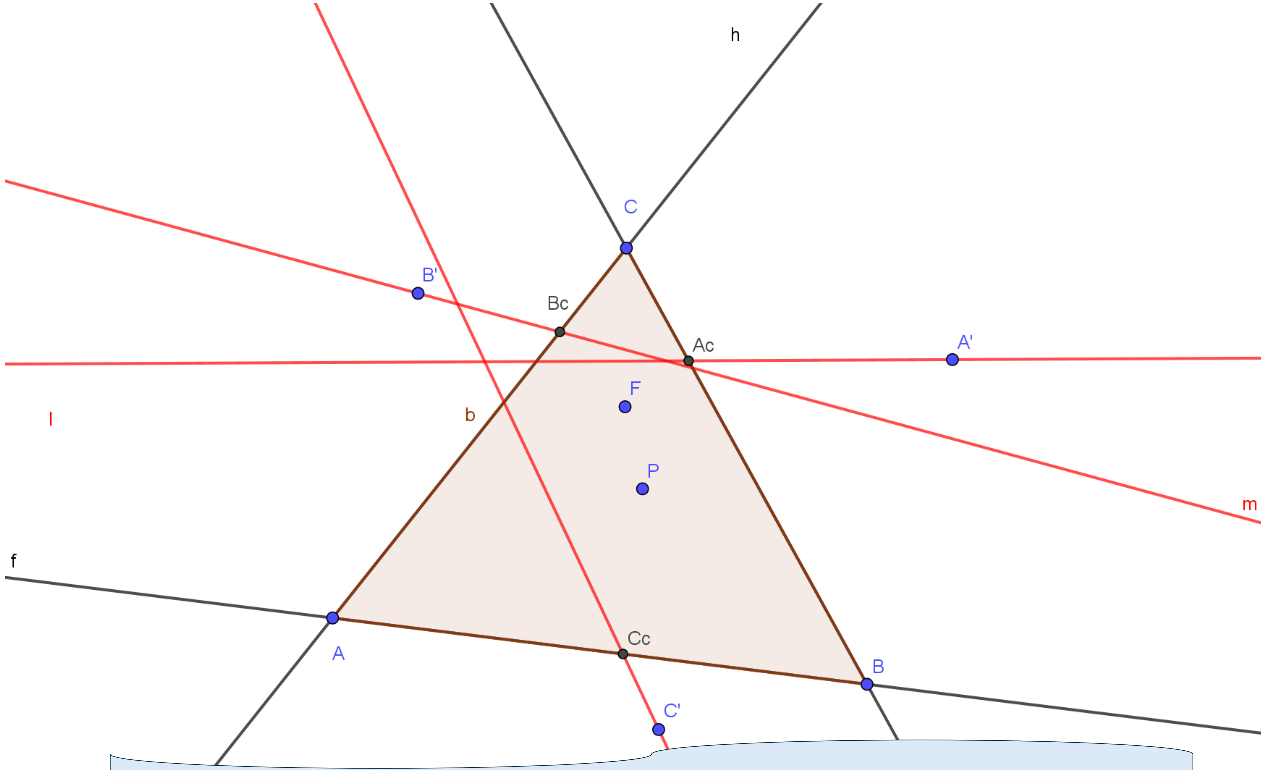


Figure 4: Construction of Garcia Capitan's example

See http://garciacapitan.epizy.com and the generalization of a problema by D. Nguyen, from April 2025. Telegram group "Geometría del triángulo".
https://garciacapitan.blogspot.com/2025/04/generalizing-problem-nguyen085.html

GeoGebra's Answer

The thesis is the locus of point F such that the lines defined by A'Ac, B'Bc and C'Cc are concurrent

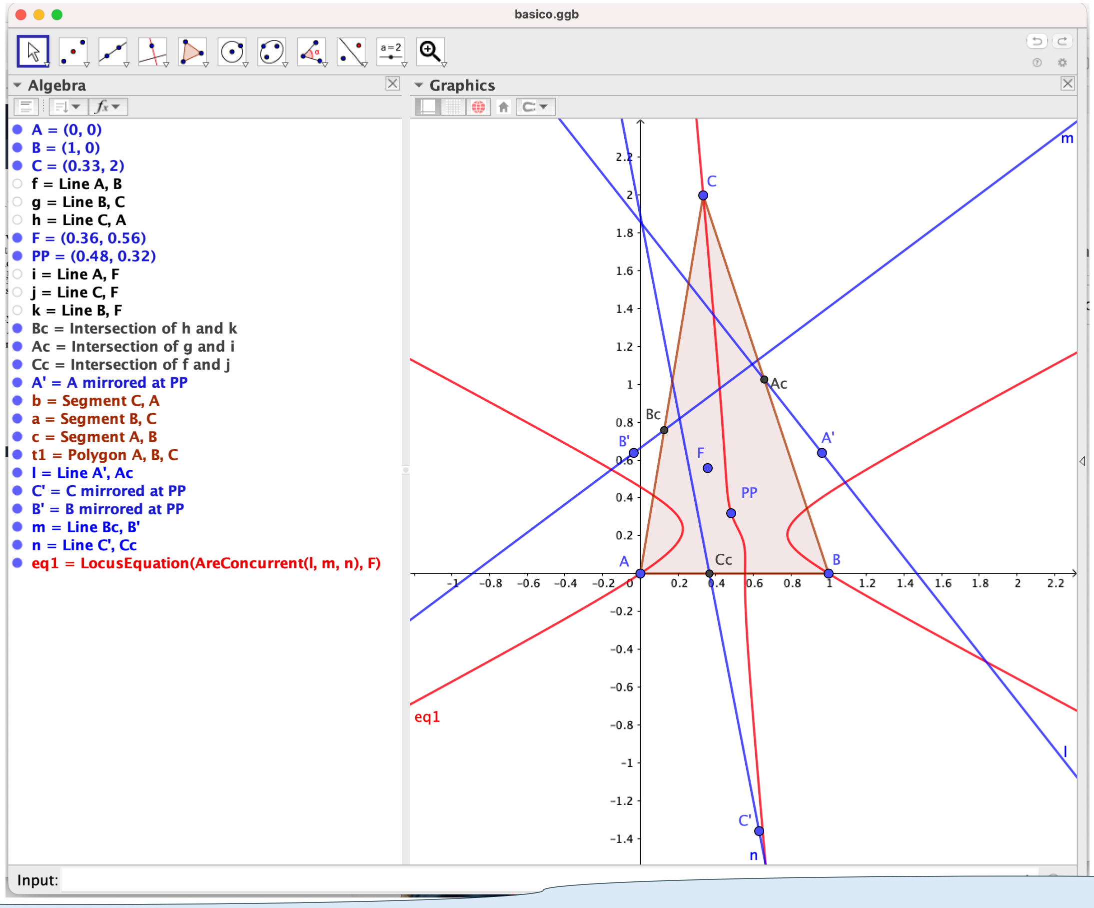


Figure 5: GeoGebra Discovery output of `LocusEquation(AreConcurrent(m,l,n),F)`

The Role of Maple

Construction ideal
Hypotheses

w.l.o.g
A(0,0)
B(1,0)

```
> Prepre:=<f1*v - f2*u, c1*v - c2*u + c2 - v,f1*q - f2*p + f2 - q
  , c1*q - c2*p,c1*f2 - c1*s - c2*f1 + c2*r + f1*s - f2*r, s,aa1-2*
  pp1, aa2-2*pp2,bb1-(2*pp1-1),bb2-2*pp2,cc1-(2*pp1-c1), cc2-(2*pp2
  -c2), c2*t-1>;
```

$$Prepre := \langle s, aa1 - 2\,pp1, aa2 - 2\,pp2, bb2 - 2\,pp2, bb1 - 2\,pp1 + 1, cc1 - 2\,pp1 + c1, cc2 - 2\,pp2 + c2, c2\,t - 1, c1\,q - c2\,p, f1\,v - f2\,u, c1\,v - c2\,u + c2 - v, f1\,q - f2\,p + f2 - q, c1\,f2 - c1\,s - c2\,f1 + c2\,r + f1\,s - f2\,r\rangle \tag{5.4}$$

```
> HilbertDimension(Prepre);
```

$$6 \tag{5.5}$$

```
> EliminationIdeal(Prepre,{c1,c2,f1,f2,pp1,pp2});
```

$$\langle 0\rangle \tag{5.6}$$

C,F and P
free

We add the condition of concurrency of the three lines

```
> Condcond:=<f1*v - f2*u, c1*v - c2*u + c2 - v,f1*q - f2*p + f2 -
  q , c1*q - c2*p,c1*f2 - c1*s - c2*f1 + c2*r + f1*s - f2*r, s,aa1
  -2*pp1, aa2-2*pp2,bb1-(2*pp1-1),bb2-2*pp2,cc1-(2*pp1-c1), cc2-(2*
  pp2-c2),aa1*v - aa1*y - aa2*u + aa2*x + u*y - v*x, bb1*q - bb1*y
  - bb2*p + bb2*x + p*y - q*x,cc1*s - cc1*y - cc2*r + cc2*x + r*y -
  s*x,c2*t-1>;
```

Hypotheses + Thesis

Parameters and Specialization

Eliminate over F, C and P

```
> HilbertDimension(Condcond);
```

$$5 \tag{5.10}$$

```
> EliminationIdeal(Condcond,{c1,c2,f1,f2,pp1});
```

$$\langle 0 \rangle \tag{5.11}$$

```
> EliminationIdeal(Condcond,{c1,c2,f1,f2,pp1,pp2});
```

$$\begin{aligned}
\langle &-2\,\mathit{pp2}\,\mathit{f2}^3\,\mathit{c1}^3 + 2\,\mathit{pp2}^2\,\mathit{f2}^2\,\mathit{c1}^3 + 4\,\mathit{pp2}\,\mathit{f1}\,\mathit{f2}^2\,\mathit{c1}^2\,\mathit{c2} + 2\,\mathit{f2}^3\,\mathit{pp1}\,\mathit{c1}^2\,\mathit{c2} \\
&- 6\,\mathit{pp2}\,\mathit{f2}^2\,\mathit{pp1}\,\mathit{c1}^2\,\mathit{c2} - 6\,\mathit{pp2}^2\,\mathit{f1}\,\mathit{f2}^2\,\mathit{c1}^2 + 6\,\mathit{pp2}\,\mathit{f2}^3\,\mathit{pp1}\,\mathit{c1}^2 - 2\,\mathit{pp2}\,\mathit{f1}^2\,\mathit{f2}\,\mathit{c1}\,\mathit{c2}^2 \\
&- 2\,\mathit{pp2}^2\,\mathit{f1}^2\,\mathit{c1}\,\mathit{c2}^2 - 4\,\mathit{f1}\,\mathit{f2}^2\,\mathit{pp1}\,\mathit{c1}\,\mathit{c2}^2 + 4\,\mathit{pp2}\,\mathit{f1}\,\mathit{f2}\,\mathit{pp1}\,\mathit{c1}\,\mathit{c2}^2 + 4\,\mathit{f2}^2\,\mathit{pp1}^2\,\mathit{c1}\,\mathit{c2}^2 \\
&+ 8\,\mathit{pp2}^2\,\mathit{f1}^2\,\mathit{f2}\,\mathit{c1}\,\mathit{c2} - 4\,\mathit{pp2}\,\mathit{f1}\,\mathit{f2}^2\,\mathit{pp1}\,\mathit{c1}\,\mathit{c2} - 4\,\mathit{f2}^3\,\mathit{pp1}^2\,\mathit{c1}\,\mathit{c2} + 2\,\mathit{f1}^2\,\mathit{f2}\,\mathit{pp1}\,\mathit{c2}^3 \\
&+ 2\,\mathit{pp2}\,\mathit{f1}^2\,\mathit{pp1}\,\mathit{c2}^3 - 4\,\mathit{f1}\,\mathit{f2}\,\mathit{pp1}^2\,\mathit{c2}^3 - 2\,\mathit{pp2}^2\,\mathit{f1}^3\,\mathit{c2}^2 - 2\,\mathit{pp2}\,\mathit{f1}^2\,\mathit{f2}\,\mathit{pp1}\,\mathit{c2}^2 \\
&+ 4\,\mathit{f1}\,\mathit{f2}^2\,\mathit{pp1}^2\,\mathit{c2}^2 - \mathit{f2}^3\,\mathit{c1}^2\,\mathit{c2} + \mathit{pp2}\,\mathit{f2}^2\,\mathit{c1}^2\,\mathit{c2} + 2\,\mathit{f1}\,\mathit{f2}^2\,\mathit{c1}\,\mathit{c2}^2 + 2\,\mathit{pp2}^2\,\mathit{f1}\,\mathit{c1}\,\mathit{c2}^2 \\
&- 2\,\mathit{f2}^2\,\mathit{pp1}\,\mathit{c1}\,\mathit{c2}^2 - 2\,\mathit{pp2}\,\mathit{f2}\,\mathit{pp1}\,\mathit{c1}\,\mathit{c2}^2 - 2\,\mathit{pp2}\,\mathit{f1}\,\mathit{f2}^2\,\mathit{c1}\,\mathit{c2} - 8\,\mathit{pp2}^2\,\mathit{f1}\,\mathit{f2}\,\mathit{c1}\,\mathit{c2} \\
&+ 2\,\mathit{f2}^3\,\mathit{pp1}\,\mathit{c1}\,\mathit{c2} + 8\,\mathit{pp2}\,\mathit{f2}^2\,\mathit{pp1}\,\mathit{c1}\,\mathit{c2} + 6\,\mathit{pp2}^2\,\mathit{f1}\,\mathit{f2}^2\,\mathit{c1} - 6\,\mathit{pp2}\,\mathit{f2}^3\,\mathit{pp1}\,\mathit{c1} \\
&- \mathit{f1}^2\,\mathit{f2}\,\mathit{c2}^3 - \mathit{pp2}\,\mathit{f1}^2\,\mathit{c2}^3 + 2\,\mathit{f1}\,\mathit{f2}\,\mathit{pp1}\,\mathit{c2}^3 - 2\,\mathit{pp2}\,\mathit{f1}\,\mathit{pp1}\,\mathit{c2}^3 + 2\,\mathit{f2}\,\mathit{pp1}^2\,\mathit{c2}^3 \\
&+ 2\,\mathit{pp2}\,\mathit{f1}^2\,\mathit{f2}\,\mathit{c2}^2 + 4\,\mathit{pp2}^2\,\mathit{f1}^2\,\mathit{c2}^2 - 2\,\mathit{f1}\,\mathit{f2}^2\,\mathit{pp1}\,\mathit{c2}^2 - 4\,\mathit{f2}^2\,\mathit{pp1}^2\,\mathit{c2}^2 \\
&- 4\,\mathit{pp2}^2\,\mathit{f1}^2\,\mathit{f2}\,\mathit{c2} + 2\,\mathit{pp2}\,\mathit{f1}\,\mathit{f2}^2\,\mathit{pp1}\,\mathit{c2} + 2\,\mathit{f2}^3\,\mathit{pp1}^2\,\mathit{c2} - \mathit{f2}^3\,\mathit{c1}\,\mathit{c2} + \mathit{pp2}\,\mathit{f2}^2\,\mathit{c1}\,\mathit{c2} \\
&+ 2\,\mathit{pp2}\,\mathit{f2}^3\,\mathit{c1} - 2\,\mathit{pp2}^2\,\mathit{f2}^2\,\mathit{c1} + \mathit{pp2}\,\mathit{f1}\,\mathit{c2}^3 - \mathit{f2}\,\mathit{pp1}\,\mathit{c2}^3 + \mathit{f1}\,\mathit{f2}^2\,\mathit{c2}^2 - 2\,\mathit{pp2}\,\mathit{f1}\,\mathit{f2}\,\mathit{c2}^2 \\
&- 2\,\mathit{pp2}^2\,\mathit{f1}\,\mathit{c2}^2 + \mathit{f2}^2\,\mathit{pp1}\,\mathit{c2}^2 + 2\,\mathit{pp2}\,\mathit{f2}\,\mathit{pp1}\,\mathit{c2}^2 - 2\,\mathit{pp2}\,\mathit{f1}\,\mathit{f2}^2\,\mathit{c2} + 4\,\mathit{pp2}^2\,\mathit{f1}\,\mathit{f2}\,\mathit{c2} \\
&- 2\,\mathit{pp2}\,\mathit{f2}^2\,\mathit{pp1}\,\mathit{c2} \rangle
\end{aligned} \tag{5.12}$$

Cubic curve in coordinates F with C and P as coefficients

Parameters and Specialization

Verifications

```
> simplify(subs(f1=0,f2=0, -2*c1^3*f2^3*pp2 + 2*c1^3*f2^2*pp2^2 +
  4*c1^2*c2*f1*f2^2*pp2 + 2*c1^2*c2*f2^3*pp1 - 6*c1^2*c2*f2^2*pp1*
  pp2 - 6*c1^2*f1*f2^2*pp2^2 + 6*c1^2*f2^3*pp1*pp2 - 2*c1*c2^2*
  f1^2*f2*pp2 - 2*c1*c2^2*f1^2*pp2^2 - 4*c1*c2^2*f1*f2^2*pp1 + 4*
  c1*c2^2*f1*f2*pp1*pp2 + 4*c1*c2^2*f2^2*pp1^2 + 8*c1*c2*f1^2*f2*
  pp2^2 - 4*c1*c2*f1*f2^2*pp1*pp2 - 4*c1*c2*f2^3*pp1^2 + 2*c2^3*
  f1^2*f2*pp1 + 2*c2^3*f1^2*pp1*pp2 - 4*c2^3*f1*f2*pp1^2 - 2*c2^2*
  f1^3*pp2^2 - 2*c2^2*f1^2*f2*pp1*pp2 + 4*c2^2*f1*f2^2*pp1^2 -
  c1^2*c2*f2^3 + c1^2*c2*f2^2*pp2 + 2*c1*c2^2*f1*f2^2 + 2*c1*c2^2*
  f1*pp2^2 - 2*c1*c2^2*f2^2*pp1 - 2*c1*c2^2*f2*pp1*pp2 - 2*c1*c2*
  f1*f2^2*pp2 - 8*c1*c2*f1*f2*pp2^2 + 2*c1*c2*f2^3*pp1 + 8*c1*c2*
  f2^2*pp1*pp2 + 6*c1*f1*f2^2*pp2^2 - 6*c1*f2^3*pp1*pp2 - c2^3*
  f1^2*f2 - c2^3*f1^2*pp2 + 2*c2^3*f1*f2*pp1 - 2*c2^3*f1*pp1*pp2 +
  2*c2^3*f2*pp1^2 + 2*c2^2*f1^2*f2*pp2 + 4*c2^2*f1^2*pp2^2 - 2*
  c2^2*f1*f2^2*pp1 - 4*c2^2*f2^2*pp1^2 - 4*c2*f1^2*f2*pp2^2 + 2*c2*
  f1*f2^2*pp1*pp2 + 2*c2*f2^3*pp1^2 - c1*c2*f2^3 + c1*c2*f2^2*pp2 +
  2*c1*f2^3*pp2 - 2*c1*f2^2*pp2^2 + c2^3*f1*pp2 - c2^3*f2*pp1 +
  c2^2*f1*f2^2 - 2*c2^2*f1*f2*pp2 - 2*c2^2*f1*pp2^2 + c2^2*f2^2*pp1
  + 2*c2^2*f2*pp1*pp2 - 2*c2*f1*f2^2*pp2 + 4*c2*f1*f2*pp2^2 - 2*c2*
  f2^2*pp1*pp2));
                                   0                                 (5.14)
```

A is in the cubic

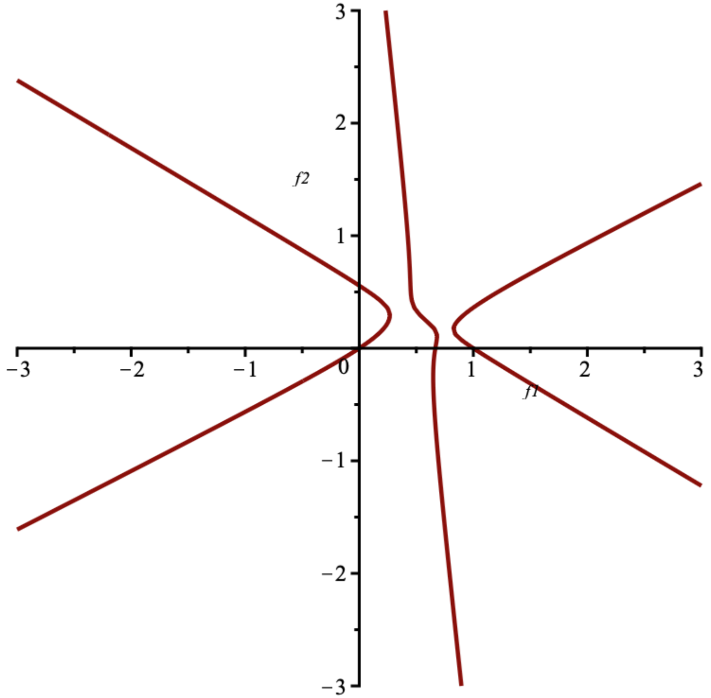
```
> simplify(subs(f1=1,f2=0, -2*c1^3*f2^3*pp2 + 2*c1^3*f2^2*pp2^2 +
  4*c1^2*c2*f1*f2^2*pp2 + 2*c1^2*c2*f2^3*pp1 - 6*c1^2*c2*f2^2*pp1*
  pp2 - 6*c1^2*f1*f2^2*pp2^2 + 6*c1^2*f2^3*pp1*pp2 - 2*c1*c2^2*
  f1^2*f2*pp2 - 2*c1*c2^2*f1^2*pp2^2 - 4*c1*c2^2*f1*f2^2*pp1 + 4*
  c1*c2^2*f1*f2*pp1*pp2 + 4*c1*c2^2*f2^2*pp1^2 + 8*c1*c2*f1^2*f2*
  pp2^2 - 4*c1*c2*f1*f2^2*pp1*pp2 - 4*c1*c2*f2^3*pp1^2 + 2*c2^3*
  f1^2*f2*pp1 + 2*c2^3*f1^2*pp1*pp2 - 4*c2^3*f1*f2*pp1^2 - 2*c2^2*
  f1^3*pp2^2 - 2*c2^2*f1^2*f2*pp1*pp2 + 4*c2^2*f1*f2^2*pp1^2 -
  c1^2*c2*f2^3 + c1^2*c2*f2^2*pp2 + 2*c1*c2^2*f1*f2^2 + 2*c1*c2^2*
  f1*pp2^2 - 2*c1*c2^2*f2^2*pp1 - 2*c1*c2^2*f2*pp1*pp2 - 2*c1*c2*
  f1*f2^2*pp2 - 8*c1*c2*f1*f2*pp2^2 + 2*c1*c2*f2^3*pp1 + 8*c1*c2*
  f2^2*pp1*pp2 + 6*c1*f1*f2^2*pp2^2 - 6*c1*f2^3*pp1*pp2 - c2^3*
  f1^2*f2 - c2^3*f1^2*pp2 + 2*c2^3*f1*f2*pp1 - 2*c2^3*f1*pp1*pp2 +
  2*c2^3*f2*pp1^2 + 2*c2^2*f1^2*f2*pp2 + 4*c2^2*f1^2*pp2^2 - 2*
  c2^2*f1*f2^2*pp1 - 4*c2^2*f2^2*pp1^2 - 4*c2*f1^2*f2*pp2^2 + 2*c2*
  f1*f2^2*pp1*pp2 + 2*c2*f2^3*pp1^2 - c1*c2*f2^3 + c1*c2*f2^2*pp2 +
  2*c1*f2^3*pp2 - 2*c1*f2^2*pp2^2 + c2^3*f1*pp2 - c2^3*f2*pp1 +
  c2^2*f1*f2^2 - 2*c2^2*f1*f2*pp2 - 2*c2^2*f1*pp2^2 + c2^2*f2^2*pp1
  + 2*c2^2*f2*pp1*pp2 - 2*c2*f1*f2^2*pp2 + 4*c2*f1*f2*pp2^2 - 2*c2*
  f2^2*pp1*pp2));
                                   0                                 (5.15)
```

B is in the cubic

Parameters and Specialization

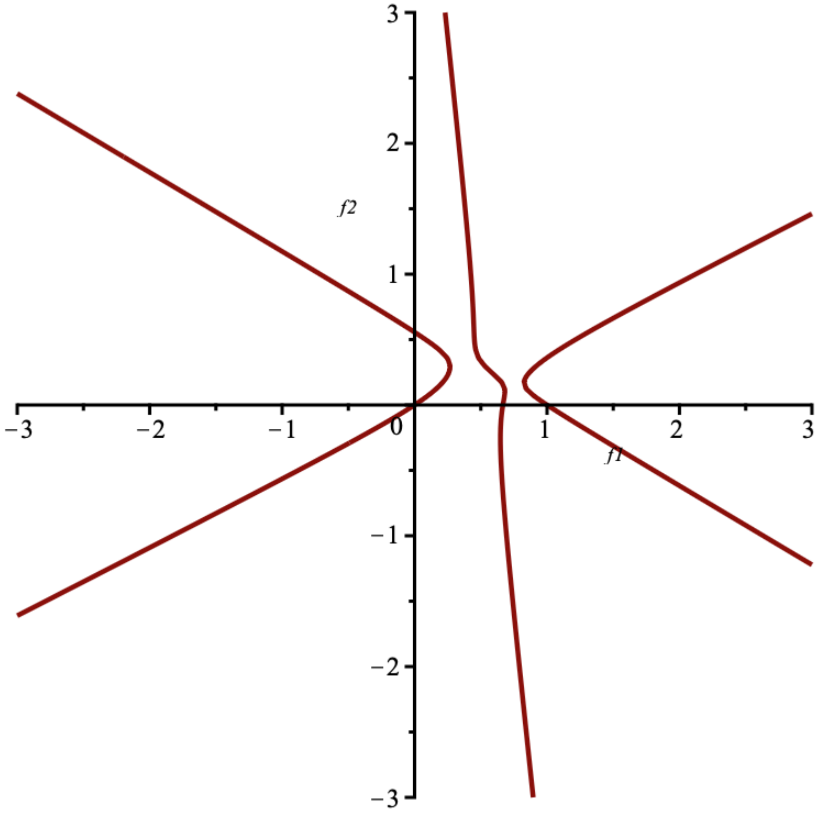
To plot the cubic, we specialize to $C = \left(\frac{1}{3}, 2\right)$ and $P = \left(\frac{1}{2}, \frac{1}{3}\right)$

```
> with(plots, implicitplot):implicitplot(216*f1^3 - 36*f1^2*f2 -
  702*f1*f2^2 - 75*f2^3 - 360*f1^2 + 360*f1*f2 + 430*f2^2 + 144*f1
  - 216*f2,f1=-3..3,f2=-3..3);
```

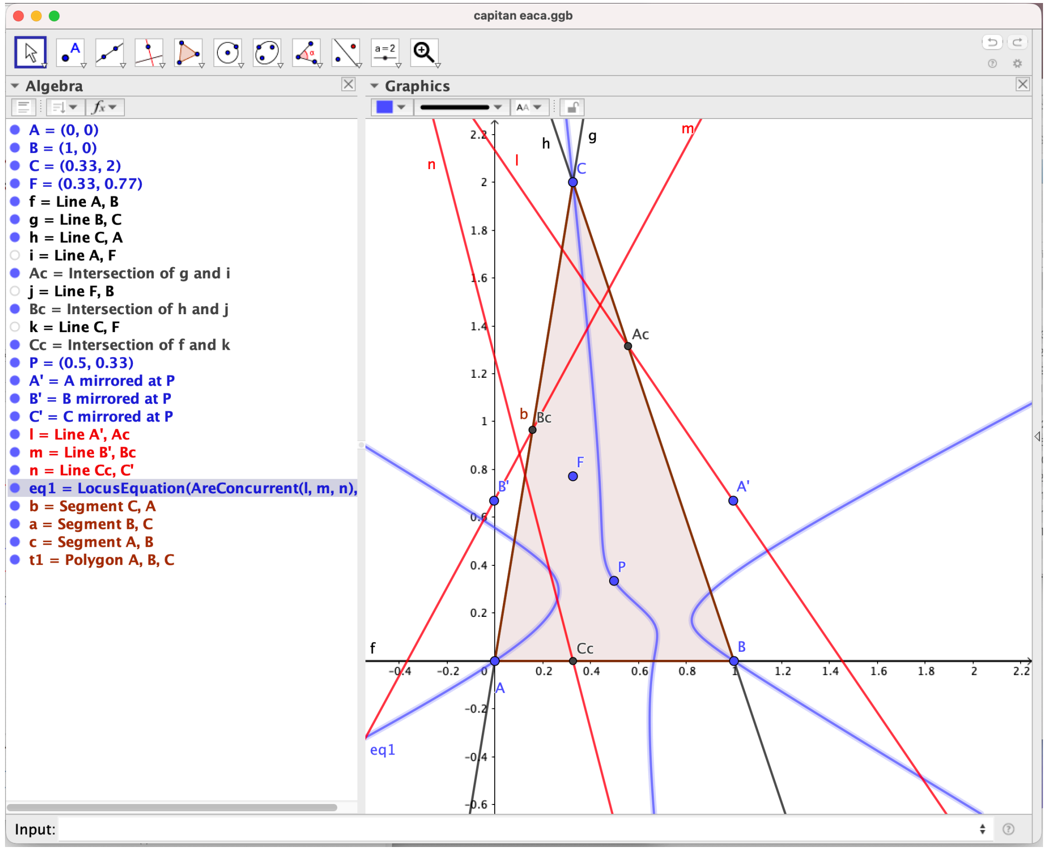


Equivalent Results

Specialization for $\mathrm{C} = \left(\frac{1}{3}, 2\right)$ and $\mathrm{P} = \left(\frac{1}{2}, \frac{1}{3}\right)$



Maple.
Elimination and Specialization



GeoGebra.
Specialization and Elimination

Can a Locus Discover a Theorem?

Cubic is a necessary condition (Hypo+Thesis⇒Cubic)

Is Cubic a sufficient condition?

To Prepre we add that A'Ac and B'Bc are concurrent at (x,y)

```
> tt:=<f1*v - f2*u, c1*v - c2*u + c2 - v,f1*q - f2*p + f2 - q , c1*q - c2*p,c1*f2 - c1*s - c2*f1 + c2*r +
  f1*s - f2*r, s,aa1-2*pp1, aa2-2*pp2,bb1-(2*pp1-1),bb2-2*pp2,cc1-(2*pp1-c1), cc2-(2*pp2-c2),aa1*v - aa1*y
  - aa2*u + aa2*x + u*y - v*x, bb1*q - bb1*y - bb2*p + bb2*x + p*y - q*x,c2*t-1>;
```

$$tt := \langle s, aa1 - 2\,pp1, aa2 - 2\,pp2, bb2 - 2\,pp2, bb1 - 2\,pp1 + 1, cc1 - 2\,pp1 + c1, cc2 - 2\,pp2 + c2, c2\,t - 1, c1\,q - c2\,p, f1\,v - f2\,u, c1\,v - c2\,u + c2 - v, f1\,q - p\,f2 + f2 - q, aa1\,v - aa1\,y - aa2\,u + aa2\,x + u\,y - v\,x, bb1\,q - bb1\,y - bb2\,p + bb2\,x + p\,y - q\,x, f2\,c1 - c1\,s - c2\,f1 + c2\,r + f1\,s - f2\,r\rangle \qquad (6.1)$$

```
> HilbertDimension(tt);
```

$$6 \qquad (6.2)$$

```
> EliminationIdeal(tt,{c1,c2,f1,f2,pp1,pp2});
```

$$\langle 0\rangle \qquad (6.3)$$

And hh is hypothesis tt plus cubic of dimension 5 and with c1,c2,pp1,pp2 and f1 as free variables

```
> HilbertDimension(hh);
```

$$5 \qquad (6.5)$$

```
> EliminationIdeal(hh,{c1,c2,pp1,pp2,f1});
```

$$\langle 0\rangle \qquad (6.6)$$

Discovering Geometric Statements

We construct JT = ⟨Hypothesis,Cubic,¬Thesis⟩ and eliminate over the independent variables

```
> EliminationIdeal(JT,{c1,c2,pp1,pp2,f1});
  factor(op(1,%))
```

$$-f1\, pp2\, (f1-1)\, (c1-f1)\, (2\, c1\, pp2 - 2\, pp1\, c2 + c2 - 2\, pp2)\, (2\, c1\, pp2 - 2\, pp1\, c2 + c2)\, (2\, c1\, pp2 - 2\, c2\, f1 + 2\, f1\, pp2 + c2 - 2\, pp2) \quad (6.9)$$

Cubic is a sufficient condition, except in certain degenerate cases:

- f 1 = 0 (F aligned with AB)
- pp2 = 0 (PP aligned with AB)
- f1 = 1 (first coordinates of F and B are equal)
- c1 = f 1
- 2c1pp2-2c2pp1+c2-2pp2 (C, 2PP and B are aligned)
- −c1 ∗ (0−2 ∗ pp2)+c2 ∗ (1−2 ∗ pp1) (CA is perpendicular to the perpendicular to (B,-2PP))
- 2 ∗ c1 ∗ pp2−2 ∗ f 1 ∗ c2+2 ∗ f 1 ∗ pp2+c2−2 ∗ pp2 = 0

Discovering Geometric Statements

This happens many times, necessary becomes sufficient, but not always

Hypothesis

Thesis

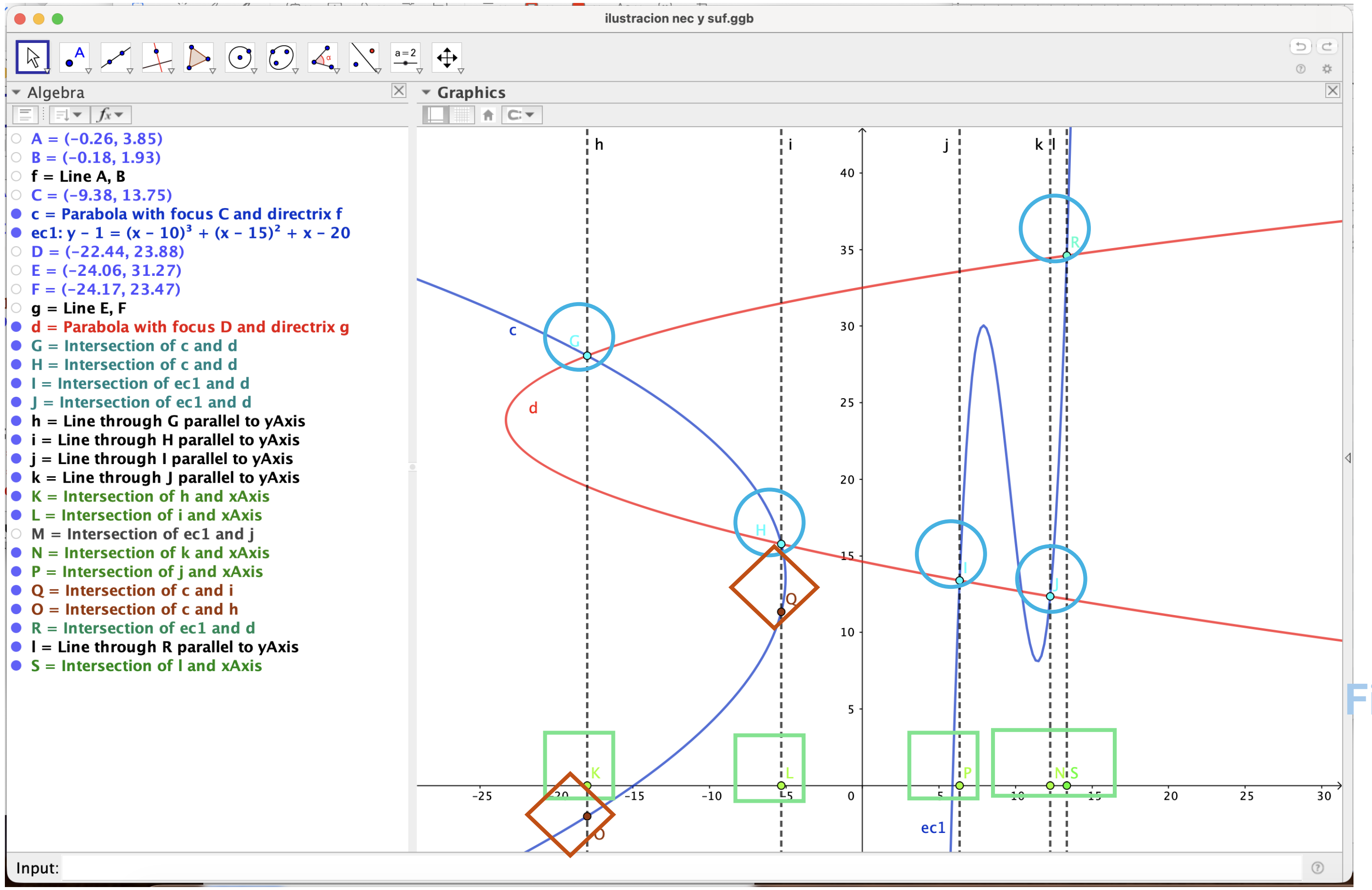
Hypothesis + Thesis

Five points

Locus points

Necessary condition

Five points



Hypothesis

+

Locus points $\subset R^2$

Not sufficient

Five points + Two points

Recio T. , Vélez, M.P.: Automatic Discovery of Theorems in Elementary Geometry. Journal of Automated Reasoning, 23(1), 63–82, 1999

Kovács, Z., Recio, T., Vélez, M.P. Detecting truth, just on parts. Revista Matemática Complutense, Volume 32, Issue 2, May 2019. 451-474

Take-Home Messages

Symbolic locus computation remains a challenging open problem

- Not a ready-to-implement solution for DG systems
- An exploratory step toward symbolic locus computation
- Incremental approach:
  - Identify where symbolic computation adds value
  - Analyze computational challenges using a CAS (Maple)
  - Extract ideas for future DG tools
- Maple serves as an effective laboratory for exploring symbolic approaches
- Ongoing work towards more general and interpretable methods

Thank you for your attention

Any questions or comments?